

## UPPER BOUNDS FOR PARTIAL SPREADS

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**ABSTRACT.** A *partial  $t$ -spread* in  $\mathbb{F}_q^n$  is a collection of  $t$ -dimensional subspaces with trivial intersection such that each non-zero vector is covered at most once. We present some improved upper bounds on the maximum sizes.

**Keywords:** Galois geometry, partial spreads, constant dimension codes, and vector space partitions

**MSC:** 51E23; 05B15, 05B40, 11T71, 94B25

## 1. INTRODUCTION

Let  $q > 1$  be a prime power and  $n$  a positive integer. A *vector space partition*  $\mathcal{P}$  of  $\mathbb{F}_q^n$  is a collection of subspaces with the property that every non-zero vector is contained in a unique member of  $\mathcal{P}$ . If  $\mathcal{P}$  contains  $m_d$  subspaces of dimension  $d$ , then  $\mathcal{P}$  is of type  $k^{m_k} \dots 1^{m_1}$ . We may leave out some of the cases with  $m_d = 0$ . Subspaces of dimension 1 are called *holes*. If there is at least one non-hole, then  $\mathcal{P}$  is called non-trivial.

A *partial  $t$ -spread* in  $\mathbb{F}_q^n$  is a collection of  $t$ -dimensional subspaces such that the non-zero vectors are covered at most once, i.e., a vector space partition of type  $t^{m_t} 1^{m_1}$ . By  $A_q(n, 2t; t)$  we denote the maximum value of  $m_t$ <sup>1</sup>. Writing  $n = kt + r$ , with  $k, r \in \mathbb{N}_0$  and  $r \leq t - 1$ , we can state that for  $r \leq 1$  or  $n \leq 2t$  the exact value of  $A_q(n, 2t; t)$  was known for more than forty years [1]. Via a computer search the cases  $A_2(3k + 2, 6; 3)$  were settled in 2010 by El-Zanati et al. [5]. In 2015 the case  $q = r = 2$  was resolved by continuing the original approach of Beutelspacher [13], i.e., by *considering* the set of holes in  $(n - 2)$ -dimensional subspaces and some averaging arguments. Very recently, Năstase and Sissokho found a very clear generalized averaging method for the number of holes in  $(n - j)$ -dimensional subspaces, where  $j \leq t - 2$ , and general  $q$ , see [14]. Their Theorem 5 determines the exact values of  $A_q(kt + r, 2t; t)$  in all cases where  $t > \frac{[r]_q}{q-1} := \frac{q^r - 1}{q - 1}$ . Here, we streamline and generalize their approach leading to improved upper bounds on  $A_q(n, 2t; t)$ , c.f. [15].

## 2. SUBSPACES WITH THE MINIMUM NUMBER OF HOLES

**Definition 2.1.** A vector space partition  $\mathcal{P}$  of  $\mathbb{F}_q^n$  has *hole-type*  $(t, s, m_1)$ , if it is of type  $t^{m_t} \dots s^{m_s} 1^{m_1}$ , for some integers  $n > t \geq s \geq 2$ ,  $m_i \in \mathbb{N}_0$  for  $i \in \{1, s, \dots, t\}$ , and  $\mathcal{P}$  is non-trivial.

**Lemma 2.2.** (C.f. [14, Proof of Lemma 9].) Let  $\mathcal{P}$  be a non-trivial vector space partition of  $\mathbb{F}_q^n$  of hole-type  $(t, s, m_1)$  and  $l, x \in \mathbb{N}_0$  with  $\sum_{i=s}^t m_i = lq^s + x$ .  $\mathcal{P}_H = \{U \cap H : U \in \mathcal{P}\}$  is a vector space partition of type  $t^{m'_t} \dots (s-1)^{m'_{s-1}} 1^{m'_1}$ , for a hyperplane  $H$  with  $\widehat{m}_1$  holes (of  $\mathcal{P}$ ). We have  $\widehat{m}_1 \equiv \frac{m_1 + x - 1}{q} \pmod{q^{s-1}}$ . If  $s > 2$ , then  $\mathcal{P}_H$  is non-trivial and  $m'_1 = \widehat{m}_1$ .

**PROOF.** If  $U \in \mathcal{P}$ , then  $\dim(U) - \dim(U \cap H) \in \{0, 1\}$  for an arbitrary hyperplane  $H$ . Since  $\mathcal{P}$  is non-trivial, we have  $n \geq s$ . For  $s > 2$ , counting the 1-dimensional subspaces of  $\mathbb{F}_q^n$  and  $H$ , via  $\mathcal{P}$  and  $\mathcal{P}_H$ , yields

$$(lq^s + x) \cdot \begin{bmatrix} s \\ 1 \end{bmatrix}_q + aq^s + m_1 = \begin{bmatrix} n \\ 1 \end{bmatrix}_q \quad \text{and} \quad (lq^s + x) \cdot \begin{bmatrix} s-1 \\ 1 \end{bmatrix}_q + a'q^{s-1} + \widehat{m}_1 = \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q$$

for some  $a, a' \in \mathbb{N}_0$ . Since  $1 + q \cdot \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q - \begin{bmatrix} n \\ 1 \end{bmatrix}_q = 0$  we conclude  $1 + q\widehat{m}_1 - m_1 - x \equiv 0 \pmod{q^s}$ . Thus,  $\mathbb{Z} \ni \widehat{m}_1 \equiv \frac{m_1 + x - 1}{q} \pmod{q^{s-1}}$ . For  $s = 2$  we have

$$(lq^2 + x) \cdot (q + 1) + aq^2 + m_1 = \begin{bmatrix} n \\ 1 \end{bmatrix}_q \quad \text{and} \quad (lq^2 + x) \cdot 1 + a'q + \widehat{m}_1 = \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q$$

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<sup>1</sup>The more general notation  $A_q(n, 2t - 2w; t)$  denotes the maximum cardinality of a collection of  $t$ -dimensional subspaces, whose pairwise intersections have a dimension of at most  $w$ . Those objects are called *constant dimension codes*, see e.g. [6]. For known bounds, we refer to <http://subspacecodes.uni-bayreuth.de> [10] containing also the generalization to *subspace codes* of mixed dimension.

leading to the same conclusion  $\widehat{m}_1 \equiv \frac{m_1+x-1}{q} \pmod{q^{s-1}}$ .  $\square$

**Lemma 2.3.** (C.f. [14, Proof of Lemma 9].) *Let  $\mathcal{P}$  be a vector space partition of  $\mathbb{F}_q^n$  of hole-type  $(t, s, m_1)$ ,  $l, x \in \mathbb{N}_0$  with  $\sum_{i=s}^t m_i = lq^s + x$ , and  $b, c \in \mathbb{Z}$  with  $m_1 = bq^s + c \geq 1$ . If  $x \geq 1$ , then there exists a hyperplane  $\widehat{H}$  with  $\widehat{m}_1 = \widehat{b}q^{s-1} + \widehat{c}$  holes, where  $\widehat{c} := \frac{c+x-1}{q} \in \mathbb{Z}$  and  $b > \widehat{b} \in \mathbb{Z}$ .*

PROOF. Apply Lemma 2.2 and observe  $m_1 \equiv c \pmod{q^s}$ . Let the number of holes in  $\widehat{H}$  be minimal. Then,

$$\widehat{m}_1 \leq \text{average number of holes per hyperplane} = m_1 \cdot \frac{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q}{\begin{bmatrix} n \\ 1 \end{bmatrix}_q} < \frac{m_1}{q}. \quad (1)$$

Assuming  $\widehat{b} \geq b$  yields  $q\widehat{m}_1 \geq q \cdot (bq^{s-1} + \widehat{c}) = bq^s + c + x - 1 \geq m_1$ , which contradicts Inequality (1).  $\square$

**Corollary 2.4.** *Using the notation from Lemma 2.3, let  $\mathcal{P}$  be a non-trivial vector space partition with  $x \geq 1$  and  $f$  be the largest integer such that  $q^f$  divides  $c$ . For each  $0 \leq j \leq s - \max\{1, f\}$  there exists an  $(n-j)$ -dimensional subspace  $U$  containing  $\widehat{m}_1$  holes with  $\widehat{m}_1 \equiv \widehat{c} \pmod{q^{s-j}}$  and  $\widehat{m}_1 \leq (b-j) \cdot q^{s-j} + \widehat{c}$ , where  $\widehat{c} = \frac{c + \begin{bmatrix} j \\ 1 \end{bmatrix}_q \cdot (x-1)}{q^j}$ .*

Proof. Observe  $\widehat{m}_1 \equiv c \not\equiv 0 \pmod{q^{s-j}}$ , i.e.,  $\widehat{m}_1 \geq 1$ , for all  $j < s - f$ .  $\square$

**Lemma 2.5.** *Let  $\mathcal{P}$  be a non-trivial vector space partition of type  $t^{m_t}1^{m_1}$  of  $\mathbb{F}_q^n$  with  $m_t = lq^t + x$ , where  $l = \frac{q^{n-t}-q^r}{q^t-1}$ ,  $x \geq 2$ ,  $t = \begin{bmatrix} r \\ 1 \end{bmatrix}_q + 1 - z + u > r$ ,  $q^f | x - 1$ ,  $q^{f+1} \nmid x - 1$ , and  $f, u, z, r, x \in \mathbb{N}_0$ . For  $\max\{1, f\} \leq y \leq t$  there exists a  $(n-t+y)$ -dimensional subspace  $U$  with  $L \leq (z+y-1)q^y + w$  holes, where  $w = -(x-1)\begin{bmatrix} y \\ 1 \end{bmatrix}_q$  and  $L \equiv w \pmod{q^y}$ .*

PROOF. Apply Corollary 2.4 with  $s = t$ ,  $j = t - y$ ,  $b = \begin{bmatrix} r \\ 1 \end{bmatrix}_q$ , and  $m_1 = \begin{bmatrix} r \\ 1 \end{bmatrix}_q q^t - \begin{bmatrix} t \\ 1 \end{bmatrix}_q (x-1)$ .  $\square$

**Lemma 2.6.** *Let  $\mathcal{P}$  be a vector space partition of  $\mathbb{F}_q^n$  with  $c \geq 1$  holes and  $a_i$  denote the number of hyperplanes containing  $i$  holes. Then,  $\sum_{i=0}^c a_i = \begin{bmatrix} n \\ 1 \end{bmatrix}_q$ ,  $\sum_{i=0}^c i a_i = c \cdot \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q$  and  $\sum_{i=0}^c i(i-1) a_i = c(c-1) \cdot \begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q$ .*

PROOF. Double-count the incidences of the tuples  $(H)$ ,  $(B_1, H)$ , and  $(B_1, B_2, H)$ , where  $H$  is a hyperplane and  $B_1 \neq B_2$  are points contained in  $H$ .  $\square$

**Lemma 2.7.** *Let  $\Delta = q^{s-1}$ ,  $m \in \mathbb{Z}$ , and  $\mathcal{P}$  be a vector space partition of  $\mathbb{F}_q^n$  of hole-type  $(t, s, c)$ . Then,  $\tau_q(c, \Delta, m) \cdot \frac{q^{n-2}}{\Delta^2} - m(m-1) \geq 0$ , where*

$$\tau_q(c, \Delta, m) = m(m-1)\Delta^2 q^2 - c(2m-1)(q-1)\Delta q + c(q-1)(c(q-1)+1).$$

PROOF. Consider the three equations from Lemma 2.6.  $(c - m\Delta)(c - (m-1)\Delta)$  times the first minus  $(2c - (2m-1)\Delta - 1)$  times the second plus the third equation, and then divided by  $\Delta^2/(q-1)$ , gives

$$(q-1) \cdot \sum_{h=0}^{\lfloor c/\Delta \rfloor} (m-h)(m-h-1)a_{c-h\Delta} = \tau_q(c, \Delta, m) \cdot \frac{q^{n-2}}{\Delta^2} - m(m-1)$$

due to Lemma 2.2. Finally, we observe  $a_i \geq 0$  and  $(m-h)(m-h-1) \geq 0$  for all  $m, h \in \mathbb{Z}$ .  $\square$

**Lemma 2.8.** *For integers  $n > t \geq s \geq 2$  and  $1 \leq i \leq s-1$ , there exists no vector space partition  $\mathcal{P}$  of  $\mathbb{F}_q^n$  of hole-type  $(t, s, c)$ , where  $c = i \cdot q^s - \begin{bmatrix} s \\ 1 \end{bmatrix}_q + s - 1$ .<sup>2</sup>*

PROOF. Assume the contrary and apply Lemma 2.7 with  $m = i(q-1)$ . Setting  $y = s-1-i$  and  $\Delta = q^{s-1}$  we compute

$$\tau_q(c, \Delta, m) = -q\Delta(y(q-1)+2) + (s-1)^2 q^2 - q(s-1)(2s-5) + (s-2)(s-3).$$

Using  $y \geq 0$  we obtain  $\tau_2(c, \Delta, m) \leq s^2 + s - 2^{s+1} < 0$ . For  $s = 2$ , we have  $\tau_q(c, \Delta, m) = -q^2 + q < 0$  and for  $q, s > 2$  we have  $\tau_q(c, \Delta, m) \leq -2q^s + (s-1)^2 q^2 < 0$ . Thus, Lemma 2.7 yields a contradiction.  $\square$

<sup>2</sup>For more general non-existence results of vector space partitions see e.g. [9, Theorem 1] and the related literature.

**Theorem 2.9.** (C.f. [14, Lemma 10].) For integers  $r \geq 1$ ,  $k \geq 2$ ,  $u \geq 0$ , and  $0 \leq z \leq \lceil \frac{r}{1} \rceil_q / 2$  with  $t = \lceil \frac{r}{1} \rceil_q + 1 - z + u > r$  we have  $A_q(n, 2t; t) \leq lq^t + 1 + z(q-1)$ , where  $l = \frac{q^{n-t}-q^r}{q^t-1}$  and  $n = kt + r$ .

PROOF. Apply Lemma 2.5 with  $x = 2 + z(q-1)$  and  $y = z + 1$ . If  $z = 0$ , then  $L < 0$ . For  $z \geq 1$ , apply Lemma 2.8. Thus,  $A_q(n, 2t; t) \leq lq^t + x - 1$ .  $\square$

The known constructions for partial  $t$ -spreads give  $A_q(kt + r, 2t; t) \geq lq^t + 1$ , see e.g. [1] (or [13] for an interpretation using the more general multilevel construction for subspace codes). Thus, Theorem 2.9 is tight for  $t \geq \lceil \frac{r}{1} \rceil_q + 1$ , c.f. [14, Theorem 5].

**Theorem 2.10.** (C.f. [15, Theorem 6,7].) For integers  $r \geq 1$ ,  $k \geq 2$ ,  $y \geq \max\{r, 2\}$ ,  $z \geq 0$  with  $\lambda = q^y$ ,  $y \leq t$ ,  $t = \lceil \frac{r}{1} \rceil_q + 1 - z > r$ ,  $n = kt + r$ , and  $l = \frac{q^{n-t}-q^r}{q^t-1}$ , we have

$$A_q(n, 2t; t) \leq lq^t + \left\lceil \lambda - \frac{1}{2} - \frac{1}{2} \sqrt{1 + 4\lambda(\lambda - (z + y - 1)(q - 1) - 1)} \right\rceil.$$

PROOF. From Lemma 2.5 we conclude  $L \leq (z + y - 1)q^y - (x - 1)\lceil \frac{y}{1} \rceil_q$  and  $L \equiv -(x - 1)\lceil \frac{y}{1} \rceil_q \pmod{q^y}$  for the number of holes of a certain  $(n - t + y)$ -dimensional subspace  $U$  of  $\mathbb{F}_q^n$ .  $\mathcal{P}_U := \{P \cap U \mid P \in \mathcal{P}\}$  is of hole-type  $(t, y, L)$  if  $y \geq 2$ . Next, we will show that  $\tau_q(c, \Delta, m) \leq 0$ , where  $\Delta = q^{y-1}$  and  $c = iq^y - (x - 1)\lceil \frac{y}{1} \rceil_q$  with  $1 \leq i \leq z + y - 1$ , for suitable integers  $x$  and  $m$ . Note that, in order to apply Lemma 2.5, we have to satisfy  $x \geq 2$  and  $y \geq f$  for all integers  $f$  with  $q^f \mid x - 1$ . Applying Lemma 2.7 then gives the desired contradiction, so that  $A_q(n, 2t; t) \leq lq^t + x - 1$ .

We choose<sup>3</sup>  $m = i(q - 1) - (x - 1) + 1$ , so that  $\tau_q(c, \Delta, m) = x^2 - (2\lambda + 1)x + \lambda(i(q - 1) + 2)$ . Solving  $\tau_q(c, \Delta, m) = 0$  for  $x$  gives  $x_0 = \lambda + \frac{1}{2} \pm \frac{1}{2}\theta(i)$ , where  $\theta(i) = \sqrt{1 - 4i\lambda(q - 1) + 4\lambda(\lambda - 1)}$ . We have  $\tau_q(c, \Delta, m) \leq 0$  for  $|2x - 2\lambda - 1| \leq \theta(i)$ . We need to find an integer  $x \geq 2$  such that this inequality is satisfied for all  $1 \leq i \leq z + y - 1$ . The strongest restriction is attained for  $i = z + y - 1$ . Since  $z + y - 1 \leq \lceil \frac{r}{1} \rceil_q$  and  $u = q^y \geq q^r$ , we have  $\theta(i) \geq \theta(z + y - 1) \geq 1$ , so that  $\tau_q(c, \Delta, m) \leq 0$  for  $x = \lceil u + \frac{1}{2} - \frac{1}{2}\theta(z + y - 1) \rceil$ . (Observe  $x \leq \lambda + \frac{1}{2} + \frac{1}{2}\theta(z + y - 1)$  due to  $\theta(z + y - 1) \geq 1$ .) Since  $x \leq \lambda + 1$ , we have  $x - 1 \leq \lambda = q^y$ , so that  $q^f \mid x - 1$  implies  $f \leq y$  provided  $x \geq 2$ . The latter is true due to  $\theta(z + y - 1) \leq \sqrt{1 - 4\lambda(q - 1) + 4\lambda(\lambda - 1)} \leq \sqrt{1 + 4\lambda(\lambda - 2)} < 2(\lambda - 1)$ , which implies  $x \geq \lceil \frac{3}{2} \rceil = 2$ .

So far we have constructed a suitable  $m \in \mathbb{Z}$  such that  $\tau_q(c, \Delta, m) \leq 0$  for  $x = \lceil \lambda + \frac{1}{2} - \frac{1}{2}\theta(z + y - 1) \rceil$ . If  $\tau_q(c, \Delta, m) < 0$ , then Lemma 2.7 gives a contradiction, so that we assume  $\tau_q(c, \Delta, m) = 0$  in the following. If  $i < z + y - 1$  we have  $\tau_q(c, \Delta, m) < 0$  due to  $\theta(i) > \theta(z + y - 1)$ , so that we assume  $i = z + y - 1$ . Thus,  $\theta(z + y - 1) \in \mathbb{N}_0$ . However, we can write  $\theta(z + y - 1)^2 = 1 + 4\lambda(\lambda - (z + y - 1)(q - 1) - 1) = (2w - 1)^2 = 1 + 4w(w - 1)$  for some integer  $w$ . If  $w \notin \{0, 1\}$ , then  $\gcd(w, w - 1) = 1$ , so that either  $\lambda = q^y \mid w$  or  $\lambda = q^y \mid w - 1$ . Thus, in any case,  $w \geq q^y$ , which is impossible since  $(z + y - 1)(q - 1) \geq 1$ . Finally,  $w \in \{0, 1\}$  implies  $w(w - 1) = 0$ , so that  $\lambda - (z + y - 1)(q - 1) - 1 = 0$ . Thus,  $z + y - 1 = \lceil \frac{y}{1} \rceil_q \geq \lceil \frac{r}{1} \rceil_q$  since  $y \geq r$ . The assumptions  $y \leq t$  and  $t = \lceil \frac{r}{1} \rceil_q + 1 - z$  imply  $z + y - 1 = \lceil \frac{r}{1} \rceil_q$  and  $y = r$ . This gives  $t = r$ , which is excluded.  $\square$

Setting  $y = t$  in Theorem 2.10 yields [4, Corollary 8], which is based on [3, Theorem 1B]. And indeed, our analysis is very similar to the technique<sup>4</sup> used in [3]. Compared to [3, 4], the new ingredients essentially are lemmas 2.2 and 2.3, see also [14, Proof of Lemma 9]. [4, Corollary 8], e.g., gives  $A_2(15, 12; 6) \leq 516$ ,  $A_2(17, 14; 7) \leq 1028$ , and  $A_9(18, 16; 8) \leq 3486784442$ , while Theorem 2.10 gives  $A_2(15, 12; 6) \leq 515$ ,  $A_2(17, 14; 7) \leq 1026$ , and  $A_9(18, 16; 8) \leq 3486784420$ . Postponing the details and proofs to a more extensive and technical paper [12], we state:

- $2^4l + 1 \leq A_2(4k + 3, 8; 4) \leq 2^4l + 4$ , where  $l = \frac{2^{4k-1}-2^3}{2^4-1}$  and  $k \geq 2$ , e.g.,  $A_2(11, 8; 4) \leq 132$ ;
- $2^6l + 1 \leq A_2(6k + 4, 12; 6) \leq 2^6l + 8$ , where  $l = \frac{2^{6k-2}-2^4}{2^6-1}$  and  $k \geq 2$ , e.g.,  $A_2(16, 12; 6) \leq 1032$ ;
- $2^6l + 1 \leq A_2(6k + 5, 12; 6) \leq 2^6l + 18$ , where  $l = \frac{2^{6k-1}-2^5}{2^6-1}$  and  $k \geq 2$ , e.g.,  $A_2(17, 12; 6) \leq 2066$ ;
- $3^4l + 1 \leq A_3(4k + 3, 8; 4) \leq 3^4l + 14$ , where  $l = \frac{3^{4k-1}-3^3}{3^4-1}$  and  $k \geq 2$ , e.g.,  $A_3(11, 8; 4) \leq 2201$ ;

<sup>3</sup> Solving  $\frac{\partial \tau_q(c, \Delta, m)}{\partial m} = 0$ , i.e., minimizing  $\tau_q(c, \Delta, m)$ , yields  $m = i(q - 1) - (x - 1) + \frac{1}{2} + \frac{x-1}{q^y}$ . For  $y \geq r$  we can assume  $x - 1 < q^y$  due the known constructions for partial spreads, so that up-rounding yields the optimum integer choice. For  $y < r$  the interval  $[u + \frac{1}{2} - \frac{1}{2}\theta(i), u + \frac{1}{2} + \frac{1}{2}\theta(i)]$  may contain no integer.

<sup>4</sup> Actually, their analysis grounds on [16] and is strongly related to the classical second-order Bonferroni Inequality [2, 7, 8] in Probability Theory, see e.g. [11, Section 2.5] for another application for bounds on subspace codes.

- $3^5l + 1 \leq A_3(5k + 3, 10; 5) \leq 3^5l + 13$ , where  $l = \frac{3^{5k-2}-3^5}{3^3-1}$  and  $k \geq 2$ , e.g.,  $A_3(13, 10; 5) \leq 6574$ ;
- $3^5l + 1 \leq A_3(5k + 4, 10; 5) \leq 3^5l + 44$ , where  $l = \frac{3^{5k-1}-3^4}{3^5-1}$  and  $k \geq 2$ , e.g.,  $A_3(14, 10; 5) \leq 19727$ ;
- $3^6l + 1 \leq A_3(6k + 4, 12; 6) \leq 3^6l + 41$ , where  $l = \frac{3^{6k-2}-3^4}{3^6-1}$  and  $k \geq 2$ , e.g.,  $A_3(16, 12; 6) \leq 59090$ ;
- $3^6l + 1 \leq A_3(6k + 5, 12; 6) \leq 3^6l + 133$ , where  $l = \frac{3^{6k-1}-3^5}{3^6-1}$  and  $k \geq 2$ , e.g.,  $A_3(17, 12; 6) \leq 177280$ ;
- $3^7l + 1 \leq A_3(7k + 4, 14; 7) \leq 3^7l + 40$ , where  $l = \frac{3^{7k-3}-3^4}{3^7-1}$  and  $k \geq 2$ , e.g.,  $A_3(18, 14; 7) \leq 177187$ ;
- $4^5l + 1 \leq A_4(5k + 3, 10; 5) \leq 4^5l + 32$ , where  $l = \frac{4^{5k-2}-4^3}{4^5-1}$  and  $k \geq 2$ , e.g.,  $A_4(13, 10; 5) \leq 65568$ ;
- $4^6l + 1 \leq A_4(6k + 3, 12; 6) \leq 4^6l + 30$ , where  $l = \frac{4^{6k-3}-4^3}{4^6-1}$  and  $k \geq 2$ , e.g.,  $A_4(15, 12; 6) \leq 262174$ ;
- $4^6l + 1 \leq A_4(6k + 5, 12; 6) \leq 4^6l + 548$ , where  $l = \frac{4^{6k-1}-4^5}{4^6-1}$  and  $k \geq 2$ , e.g.,  $A_4(17, 12; 6) \leq 4194852$ ;
- $4^7l + 1 \leq A_4(7k + 4, 14; 7) \leq 4^7l + 128$ , where  $l = \frac{4^{7k-3}-4^4}{4^7-1}$  and  $k \geq 2$ , e.g.,  $A_4(18, 14; 7) \leq 4194432$ ;
- $5^5l + 1 \leq A_5(5k + 2, 10; 5) \leq 5^5l + 7$ , where  $l = \frac{5^{5k-3}-5^2}{5^5-1}$  and  $k \geq 2$ , e.g.,  $A_5(12, 10; 5) \leq 78132$ ;
- $5^5l + 1 \leq A_5(5k + 4, 10; 5) \leq 5^5l + 329$ , where  $l = \frac{5^{5k-1}-5^4}{5^5-1}$  and  $k \geq 2$ , e.g.,  $A_5(14, 10; 5) \leq 1953454$ ;
- $7^5l + 1 \leq A_7(5k + 4, 10; 5) \leq 7^5l + 1246$ , where  $l = \frac{7^{5k-1}-7^2}{7^5-1}$  and  $k \geq 2$ , e.g.,  $A_7(14, 10; 5) \leq 40354853$ ;
- $8^4l + 1 \leq A_8(4k + 3, 8; 4) \leq 8^4l + 264$ , where  $l = \frac{8^{4k-1}-8^3}{8^4-1}$  and  $k \geq 2$ , e.g.,  $A_8(11, 8; 4) \leq 2097416$ ;
- $8^5l + 1 \leq A_8(5k + 2, 10; 5) \leq 8^5l + 25$ , where  $l = \frac{8^{5k-3}-8^2}{8^5-1}$  and  $k \geq 2$ , e.g.,  $A_8(12, 10; 5) \leq 2097177$ ;
- $8^6l + 1 \leq A_8(6k + 2, 12; 6) \leq 8^6l + 21$ , where  $l = \frac{8^{6k-4}-8^2}{8^6-1}$  and  $k \geq 2$ , e.g.,  $A_8(14, 12; 6) \leq 16777237$ ;
- $9^3l + 1 \leq A_9(3k + 2, 6; 3) \leq 9^3l + 41$ , where  $l = \frac{9^{3k-1}-9^2}{9^3-1}$  and  $k \geq 2$ , e.g.,  $A_9(8, 6; 3) \leq 59090$ ;
- $9^5l + 1 \leq A_9(5k + 3, 10; 5) \leq 9^5l + 365$ , where  $l = \frac{9^{5k-2}-9^3}{9^5-1}$  and  $k \geq 2$ , e.g.,  $A_9(13, 10; 5) \leq 43047086$ ;

c.f. the web-page mentioned in footnote 1 for more numerical values and comparisons of the different upper bounds.

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